Limit of Sequences

1. Let $y = \frac{1}{1+h}$, $0 < y < 1 \Rightarrow 0 < \frac{1}{1+h} < 1 \Rightarrow 0 < h < \infty$

Since $(1 + h)^n > 1 + nh$, $\therefore 0 < \frac{1}{(1+h)^n} < \frac{1}{1+nh} \Rightarrow 0 \le \lim_{n\to\infty} \frac{1}{(1+h)^n} \le \lim_{n\to\infty} \frac{1}{1+nh} = 0$

By Sandwich Th, $\lim_{n\to\infty} y^n = \lim_{n\to\infty} \frac{1}{(1+h)^n} = 0$

Also, $(1 + h)^n > 1 + nh + \frac{n(n-1)}{2}h^2 > nh + \frac{n(n-1)}{2}h^2$

$\therefore 0 < \frac{1}{(1+h)^n} < \frac{1}{nh + \frac{n(n-1)}{2}h^2} \Rightarrow 0 \le \lim_{n\to\infty} \frac{n}{(1+h)^n} \le \lim_{n\to\infty} \frac{n}{nh + \frac{n(n-1)}{2}h^2} = \lim_{n\to\infty} \frac{1}{h + \frac{n-1}{2}h^2} = 0$

By Sandwich Th, $\lim_{n\to\infty} ny^n = \lim_{n\to\infty} \frac{n}{(1+h)^n} = 0$.

Now, consider the function, $f(x) = (1+x)^n - \frac{1}{1-nx}$, $x > 0$, $nx < 1$.

Then $f'(x) = n(1+x)^{n-1} - \frac{n}{(1-nx)^2} \Rightarrow f'(0) = 0$

and $f''(x) = n(n-1)(1+x)^{n-2} - \frac{2n^2}{(1-nx)^3} \Rightarrow f(0) = n(n-1) - 2n^2 = -n^2 - n < 0$

$\therefore$ f(x) is a max. when x = 0.

$f(0) = 0 < f(x) = (1+x)^n - \frac{1}{1-nx} \Rightarrow (1+x)^n < \frac{1}{1-nx}$. Put x = a, $(1+a)^n < \frac{1}{1-na}$.

Now, $0 < (1+x^n)^n < \frac{1}{1-nx^n}$ $(0 < x < 1) \Rightarrow 0 \le \lim_{n\to\infty}(1+x^n)^n \le \lim_{n\to\infty} \frac{1}{1-nx^n} = 0$ $\exists$ n, $nx^n < 1$ and $0 < x < 1$.

By Sandwich Th, $\lim_{n\to\infty}(1+x^n)^n = 0$.

2. $u_n - (k + k^{-1})u_{n-1} + u_{n-2} = 0$, $u_0 = 1$ …. (1)

$\Rightarrow u_n - ku_{n-1} = \frac{1}{k}(u_{n-1} - ku_{n-2}) = \frac{1}{k^2}(u_{n-2} - ku_{n-3}) = ... = \left(\frac{1}{k}\right)^{n-1}(u_1 - k)$ …. (2)

Also, from (1), $u_n - \frac{1}{k}u_{n-1} = k\left(u_{n-1} - \frac{1}{k}u_{n-1}\right) = k^2\left(u_{n-2} - \frac{1}{k}u_{n-3}\right) = ... = k^{n-1}\left(u_1 - \frac{1}{k}\right)$ …. (3)

$(3) \times k^2$, $k^2u_n - ku_{n-1} = k^{n+1}\left(u_1 - \frac{1}{k}\right)$ …. (4)

(4) – (2), $(k^2 - 1)u_n = k^{n+1}\left(u_1 - \frac{1}{k}\right) - \left(\frac{1}{k}\right)^{n-1}(u_1 - k) \Rightarrow u_n = \frac{1}{k^2 - 1}\left[k^{n+1}\left(u_1 - \frac{1}{k}\right) - \left(\frac{1}{k}\right)^{n-1}(u_1 - k)\right]$ …. (5)

The unique value of u_1 such that u_n tends to a limit as $n \to \infty$ is when

$u_1 - \frac{1}{k} = 0$ or $u_1 = \frac{1}{k}$. In this case, from (5), u_n is reduced to $u_n = \frac{1}{k^n} \to 0$ as $n \to \infty$.

3. $x_n = \sqrt{\dfrac{ab^2 + x_n{}^2}{a+1}}$, $a > 0$ and $0 < x_1 < b$. It can be easily seen that $x_n > 0$, $\forall n \in \mathbb{N}$.

Let P(n) be the proposition $x_n < b$.

P(1) is true as $0 < x_1 < b$.

Assume P(k) is true for some $k \in \mathbb{N}$, i.e. $x_k < b$ or $x_k - b < 0$ …. (1)

For P(k + 1), $x_{k+1}{}^2 - b^2 = \dfrac{ab^2 + x_n{}^2}{a+1} - b^2 = \dfrac{ab^2 + x_n{}^2 - ab^2 - b^2}{a+1} = \dfrac{x_n{}^2 - b^2}{a+1} < 0$, by (1) and $a + 1 > 0$.

$\therefore$ $x_{k+1} < b$ and P(k + 1) is true . By the Principle of Mathematical Induction, $x_n < b, \forall n \in \mathbb{N}$.

$\therefore$ x_n is bounded.

$$x_{n+1}{}^2 - x_n{}^2 = \frac{ab^2 + x_n{}^2}{a+1} - x_n{}^2 = \frac{a\left(b^2 - x_n{}^2\right)}{a+1} > 0 \quad \text{since} \quad x_n < b.$$

$\therefore x_{n+1}{}^2 < x_n{}^2 \Rightarrow x_{n+1} < x_n$ since $x_n > 0$.

$\therefore$ x_n is monotonic increasing and hence by Monotone Bound Th, limits exists.

Let $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. From $x_n = \sqrt{\dfrac{ab^2 + x_n{}^2}{a+1}}$, take $n \to \infty$, $L = \sqrt{\dfrac{ab^2 + L}{a+1}} \Rightarrow L = b$, $a \neq 0$.

4. $x_{n+1} = x_n{}^2 - 2x_n + 2$ (n > 1). $\Rightarrow$ $x_{n+1} - 1 = (x_n - 1)^2 = (x_{n-1} - 1)^{2^2} = \ldots = (x_1 - 1)^{2^{n-1}}$

(a) As $1 < x_1 < 2$, $0 < x_1 - 1 < 1$ $(x_1 - 1)^{2^{n-1}} \to 0$, as $n \to \infty$. $\therefore x_{n+1} \to 1$ or $x_n \to 1$.

(b) As $x_1 = 2$, then $(x_1 - 1)^{2^{n-1}} = 1$, $x_{n+1} - 1 \to 1$ or $x_n \to 2$.

(c) As $x_1 > 2$, $(x_1 - 1)^{2^{n-1}} \to \infty$ as $n \to \infty$. $\therefore x_{n+1} \to \infty$, $x_n \to \infty$.

5. $a_0 = a_1 \sin^2 \phi = 2 \cos \phi$ and $a_n - 2a_{n+1} + a_{n+2} \sin^2 \phi = 0$.

$$a_1 = \frac{2\cos\phi}{\sin^2\phi} = \frac{2\cos\phi}{1-\cos^2\phi} = \frac{1}{1-\cos\phi} - \frac{1}{1+\cos\phi} \quad \ldots. \quad (1)$$

$a_0 - 2a_1 + a_2 \sin^2 \phi = 0$ $\Rightarrow$ $2\cos\phi - 2\dfrac{2\cos\phi}{\sin^2\phi} + a_2 \sin^2\phi = 0$ $\quad a_2 = \dfrac{2\cos\phi}{\sin^2\phi}\left(\dfrac{2}{\sin^2\phi} - 1\right)$

$$a_1 + a_2 = \frac{4\cos\phi}{\sin^4\phi} = \frac{4\cos\phi}{\left(1-\cos^2\phi\right)^2} = \frac{4\cos\phi}{(1-\cos\phi)^2(1+\cos\phi)^2} = \left(\frac{1}{1-\cos\phi}\right)^2 - \left(\frac{1}{1+\cos\phi}\right)^2 \quad \ldots. \quad (2)$$

Let P(n) be the proposition $\displaystyle\sum_{r=1}^{n} a_r = \left(\frac{1}{1-\cos\phi}\right)^n - \left(\frac{1}{1+\cos\phi}\right)^n = \frac{1}{\alpha^n} - \frac{1}{\beta^n}$, $\begin{cases}\alpha = 1 - \cos\phi \\ \beta = 1 + \cos\phi\end{cases}$.

P(1) and P(2) are true by (1) and (2) .

Assume P(k) and P(k+1) are true for some $k \in \mathbb{N}$, i.e.

$$\sum_{r=1}^{k} a_r = \frac{1}{\alpha^k} - \frac{1}{\beta^k} \quad \ldots. \quad (3) \qquad\qquad \sum_{r=1}^{k+1} a_r = \frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}} \quad \ldots. \quad (4)$$

For P(k + 2), $a_r - 2a_{r+1} + a_{r+2} \sin^2 \phi = 0$ $\Rightarrow$ $a_r - (\alpha+\beta)a_{r+1} + \alpha\beta\, a_{r+2} = 0$

$$\therefore \sum_{r=1}^{k} a_r - (\alpha+\beta)\sum_{r=1}^{k} a_{r+1} + \alpha\beta\sum_{r=1}^{k} a_{r+2} = 0 \Rightarrow \sum_{r=1}^{k} a_r - (\alpha+\beta)\sum_{r=1}^{k+1} a_r + (\alpha+\beta)a_1 + \alpha\beta\sum_{r=1}^{k+2} a_r - \alpha\beta(a_1 + a_2) = 0$$

$$\left(\frac{1}{\alpha^k}-\frac{1}{\beta^k}\right)-(\alpha+\beta)\left(\frac{1}{\alpha^{k+1}}-\frac{1}{\beta^{k+1}}\right)+(\alpha+\beta)\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)+\alpha\beta\sum_{r=1}^{k+2} a_r-\alpha\beta\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)=0$$

On simplification, we get $\sum_{r=1}^{k+2} a_r=\frac{1}{\alpha^{k+2}}-\frac{1}{\beta^{k+2}}$. $\therefore$ P(k + 2) is true.

$\therefore$ By the Second Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbb{N}$.

6. $u_{n+1}=\frac{6u_n^{\,2}+6}{u_n^{\,2}+11}$, n = 0, 1, 2, … If u_n converges to a limit $a=\lim_{n\to\infty}u_n=\lim_{n\to\infty}u_{n+1}$.

$a=\frac{6a^2+6}{a^2+11}\Rightarrow$ $a^3 + 11a = 6a^2 + 6$ $\Rightarrow$ $a^3 – 6a^2 + 11a – 6 = 0$ $\Rightarrow$ $(a – 1)(a – 2)(a – 3) = 0$

$\therefore$ a = 1, 2, 3.

(i) If $u_0 > 3$, Assume $u_k > 3$, since $u_k > 0$, then $u_k^{\,2} > 9$ or $u_k^{\,2} – 9 > 0$ …. (1)

$$u_{k+1}-3=\frac{6u_k^{\,2}+6}{u_k^{\,2}+11}-3=\frac{6u_k^{\,2}+6-3u_k^{\,2}-33}{u_k^{\,2}+11}=\frac{3u_k^{\,2}-27}{u_k^{\,2}+11}=\frac{3(u_k^{\,2}-9)}{u_k^{\,2}+11}>0$$, by (1).

By the Principle of Mathematical Induction, $u_n > 3$, $\forall n \in \mathbb{N}$.

$$u_n-u_{n+1}=u_n-\frac{6u_n^{\,2}+6}{u_n^{\,2}+11}=\frac{u_n^{\,3}-6u_n^{\,2}+11u_n-6}{u_n^{\,2}+11}=\frac{(u_n-1)(u_n-2)(u_n-3)}{u_n^{\,2}+11}>0$$, since $u_n > 3$.

$\therefore$ $3 < u_{n+1} < u_n$.

(ii) $$\frac{9}{10}-\frac{u_{n+1}-3}{u_n-3}=\frac{9}{10}-\frac{\frac{6u_n^{\,2}+6}{u_n^{\,2}+11}-3}{u_n-3}=\frac{9}{10}-\frac{3\frac{u_n^{\,2}-9}{u_n^{\,2}+11}}{u_n-3}=\frac{9}{10}-\frac{3(u_n+3)}{u_n^{\,2}+11}=\frac{9u_n^{\,2}+99-30u_n-90}{10(u_n^{\,2}+11)}$$

$$=\frac{9u_n^{\,2}-30u_n+9}{10(u_n^{\,2}+11)}=\frac{3(3u_n-1)(u_n-3)}{10(u_n^{\,2}+11)}>0$$, since $u_n > 3$. $\therefore$ $\frac{u_{n+1}-3}{u_n-3}<\frac{9}{10}$.

From (i), u_n is monotonic decreasing and bounded below and limit exists.

From (ii), $\frac{u_{n+1}-3}{u_n-3}<\frac{9}{10}$, $\frac{u_n-3}{u_{n-1}-3}<\frac{9}{10}$, ..., $\frac{u_1-3}{u_0-3}<\frac{9}{10}$.

Multiply these inequalities, $\frac{u_{n+1}-3}{u_0-3}<\left(\frac{9}{10}\right)^{n+1}$.

But $u_n > 3$, $0<\frac{u_{n+1}-3}{u_0-3}<\left(\frac{9}{10}\right)^{n+1}\Rightarrow 0\le\lim_{n\to\infty}\frac{u_{n+1}-3}{u_0-3}\le\lim_{n\to\infty}\left(\frac{9}{10}\right)^{n+1}=0$.

By Sandwich Theorem, $\lim_{n\to\infty}\frac{u_{n+1}-3}{u_0-3}=0\Rightarrow\lim_{n\to\infty}u_{n+1}=3\Rightarrow\lim_{n\to\infty}u_n=3$.

7. $x_0 = x$, $x_{n+1}=\frac{x_n}{1+\sqrt{1+x_n^{\,2}}}$ (n = 0, 1, 2, …) , $a_n = 2^n x_n$, $b_n=\frac{2^n x_n}{\sqrt{1+x_n^{\,2}}}$

$$x_{n+1}-x_n=\frac{x_n}{1+\sqrt{1+x_n^{\,2}}}-x_n=\frac{x_n-x_n-x_n\sqrt{1+x_n^{\,2}}}{1+\sqrt{1+x_n^{\,2}}}=-\frac{x_n\sqrt{1+x_n^{\,2}}}{1+\sqrt{1+x_n^{\,2}}}$$

$\therefore$ Sign of $x_{n+1}-x_n$ depends on $–x_n$. .

(i) If $x_0 = x > 0$, then $x_n > 0$ $\forall n \in \mathbb{N}$ and $x_{n+1} - x_n < 0$.

$\therefore$ $x_{n+1} < x_n$ and x_n is monotonic decreasing.

(ii) If $x_0 = x = 0$, then $x_n = 0$ $\forall n \in \mathbb{N}$.

(iii) If $x_0 = x < 0$, then $x_n < 0 \quad \forall n \in \mathbb{N}$ and $x_{n+1} - x_n > 0$.

$\therefore$ $x_{n+1} > x_n$ and x_n is monotonic increasing.

For case **(i)** , x_n is bounded below because $x_n > 0 \quad \forall n \in \mathbb{N}$. For case **(ii)** , $x_n \to 0$ as $n \to \infty$.

For case **(iii)** , x_n is bounded above because $x_n < 0 \quad \forall n \in \mathbb{N}$.

In all cases, limit exists. Let $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$.

From $x_{n+1} = \dfrac{x_n}{1+\sqrt{1+x_n^2}}$, $L = \dfrac{L}{1+\sqrt{1+L^2}} \Rightarrow L = 0$

For the sequence $a_n = 2^n x_n$,

(i) If $x > 0$, then $0 < a_n = 2^n x_n$ and $a_{n+1} = 2^{n+1} x_{n+1} = \dfrac{2^{n+1} x_n}{1+\sqrt{1+x_n^2}} = \dfrac{2a_n}{1+\sqrt{1+x_n^2}} < \dfrac{2a_n}{1+\sqrt{1+0}} = a_n$

(ii) If $x < 0$, then $0 > a_n = 2^n x_n$ and $a_{n+1} = \dfrac{2a_n}{1+\sqrt{1+x_n^2}} > \dfrac{2a_n}{1+\sqrt{1+0}} = a_n$, since $x_n, a_n < 0$.

In both cases, a_n is monotonic and bounded. $\lim_{n\to\infty} a_n$ exists.

For the sequence b_n .

(i) If $x > 0$, then $b_n = \dfrac{2^n x_n}{\sqrt{1+x_n^2}} > 0$ since $x_n > 0$.

$$b_{n+1} = \frac{2^{n+1} x_{n+1}}{\sqrt{1+x_{n+1}^2}} = \frac{2^{n+1}}{\sqrt{1+x_{n+1}^2}}\frac{x_n}{1+\sqrt{1+x_n^2}} = 2\frac{2^n x_n}{\sqrt{1+x_n^2}}\frac{\sqrt{1+x_n^2}}{\sqrt{1+x_{n+1}^2}}\frac{1}{1+\sqrt{1+x_n^2}} < 2b_n(1)\left(\frac{1}{1+\sqrt{1+0}}\right) = b_n$$

(ii) If $x < 0$, then $b_n = \dfrac{2^n x_n}{\sqrt{1+x_n^2}} < 0$ since $x_n < 0$.

$$b_{n+1} = \frac{2^{n+1} x_{n+1}}{\sqrt{1+x_{n+1}^2}} = \frac{2^{n+1}}{\sqrt{1+x_{n+1}^2}}\frac{x_n}{1+\sqrt{1+x_n^2}} = 2\frac{2^n x_n}{\sqrt{1+x_n^2}}\frac{\sqrt{1+x_n^2}}{\sqrt{1+x_{n+1}^2}}\frac{1}{1+\sqrt{1+x_n^2}} > 2b_n(1)\left(\frac{1}{1+\sqrt{1+0}}\right) = b_n$$

In both cases, b_n is monotonic and bounded. $\lim_{n\to\infty} b_n$ exists.

Now, $b_n = \dfrac{2^n x_n}{\sqrt{1+x_n^2}} = \dfrac{a_n}{\sqrt{1+x_n^2}} \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty}\dfrac{a_n}{\sqrt{1+x_n^2}} = \dfrac{\lim_{n\to\infty} a_n}{\sqrt{1+\lim_{n\to\infty} x_n^2}} = \dfrac{\lim_{n\to\infty} a_n}{\sqrt{1+0}} = \lim_{n\to\infty} a_n$

8. $0 < a_1 < 3$ and $a_{n+1} = \dfrac{12}{1+a_n}$, $a_{n+1} - a_{n-1} = \dfrac{12}{1+a_n} - a_{n-1} = \dfrac{12}{1+\dfrac{12}{1+a_{n-1}}} - a_{n-1} = -\dfrac{(a_{n-1}-3)(a_{n-1}+4)}{a_{n-1}+13}$

Let P(n) : $0 < a_{2n-1} < 3$, $a_{2n} > 3$, $\forall n \in \mathbb{N}$.

P(1) and P(2) : $0 < a_1 < 3$, $a_2 = \dfrac{12}{1+a_1} > \dfrac{12}{1+3} = 3$

Assume $0 < a_{2k-1} < 3$, $a_{2k} > 3$, , for some $k \in \mathbb{N}$.

$$a_{2n+1} - 3 = \frac{12}{1+a_{2n}} - 3 = \frac{9-3a_{2n}}{1+a_{2n}} = \frac{3(3-a_{2n})}{1+a_{2n}} < 0, \quad a_{2n+2} - 3 = \frac{12}{1+a_{2n+1}} - 3 = \frac{9-3a_{2n+1}}{1+a_{2n+1}} = \frac{3(3-a_{2n+1})}{1+a_{2n+1}} > 0$$

$\therefore$ By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbb{N}$.

Now, $a_{2n+1} - a_{2n-1} = -\frac{(a_{2n-1}-3)(a_{2n-1}+4)}{a_{2n-1}+13} > 0$, since $a_{2n-1} < 3$.

$a_{2n} - a_{2n-2} = -\frac{(a_{2n-2}-3)(a_{2n-2}+4)}{a_{2n-2}+13} < 0$, since $a_{2n-2} > 3$.

∴ a_{2n+1} is increasing and is bounded above and a_{2n} is decreasing and is bounded below.

∴ Limits exist and let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, $a_{n+1} = \frac{12}{1+a_n} \Rightarrow L = \frac{12}{1+L} \Rightarrow L^2 + L - 12 = 0$

$\Rightarrow (L-3)(L+4) = 0 \Rightarrow L = 3$ ($L = -4$ is rejected $a_n > 0$, $\forall n \in \mathbb{N}$.)

9. Let α, β be the roots of the equation : $L^2 - L - 5 = 0$, where $\alpha < \beta$.

$\alpha + \beta = 1$, $\alpha\beta = 5$, $\alpha = \frac{1-\sqrt{21}}{2} < 0$, $\beta = \frac{1+\sqrt{21}}{2} > 0$

$u_1 = 3$, $u_{n+1} = \sqrt{u_n + 5} = \sqrt{(\alpha+\beta)u_n + \alpha\beta}$.

Let $P(n)$ be the proposition $u_n > \beta$.

$P(1)$ is true as $u_1 = 3 > \frac{1+\sqrt{21}}{2} = \beta$.

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. $u_k > \beta$ or $u_k - \beta > 0$ …. (1)

For $P(k+1)$, $u_{k+1}^2 - \beta^2 = (\alpha+\beta)u_k - \alpha\beta - \beta^2 = (\alpha+\beta)(u_k - \beta) > 0$, by (1)

∴ $u_{k+1}^2 < \beta^2$ and since $\beta > 0$, $u_{k+1} > \beta$ ∴ $P(k+1)$ is true .

By the Principle of Mathematical Induction, $P(n)$ is true , $\forall n \in \mathbb{N}$. ∴ u_n is bounded below.

Now, $u_n^2 - u_{n+1}^2 = u_n^2 - (\alpha+\beta)u_n + \alpha\beta = (u_n - \alpha)(u_n - \beta) > 0$, since $u_n > \beta > \alpha$.

∴ $u_n^2 > u_{n+1}^2$, $u_n > u_{n+1}$ and u_n is monotonic decreasing.

∴ Limit exists and let $L = \lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1}$.

$u_{n+1} = \sqrt{u_n + 5} \Rightarrow L = \sqrt{L+5} \Rightarrow L^2 - L - 5 = 0 \Rightarrow u_n \to \beta = a$ (α is rejected since $u_n > 0$ $\forall n \in \mathbb{N}$.)

In the above, $u_{2n+1}^2 - a^2 = u_{2n+1}^2 - \beta^2 = (\alpha+\beta)u_{2n} - \alpha\beta - \beta^2 = (\alpha+\beta)(u_{2n} - \beta) > 0 \Rightarrow u_{2n} > a$

$$u_{2n+1} - a = \frac{u_{2n+1}^2 - \beta^2}{u_{2n+1}+\beta} = \frac{(\alpha+\beta)(u_{2n}-\beta)}{u_{2n+1}+\beta} = \frac{u_{2n}-\beta}{u_{2n+1}+\beta} = \frac{1}{u_{2n+1}+\beta}\frac{u_{2n-1}-\beta}{u_{2n}+\beta} < \frac{1}{\beta+\beta}\frac{u_{2n-1}-\beta}{\beta+\beta} = \frac{1}{4\beta^2}(u_{2n-1}-\beta)$$

But, $\frac{1}{4\beta^2} = \frac{1}{4}\left(\frac{2}{1+\sqrt{21}}\right)^2 \approx 0.032 < \frac{1}{30}$.

∴ $u_{2n+1} - a < \frac{1}{30}(u_{2n-1} - \beta) < \frac{1}{30^2}(u_{2n-3} - \beta) < \ldots < \frac{1}{30^n}(u_1 - a)$

10. $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, $x_0 = 1$, $y_0 = 0$.

(i) $\{z_n\} = \{x_n + ay_n\}$ is geometric $\Rightarrow$ $z_n^2 = z_{n-1}z_{n+1} \Rightarrow$ $(x_n + ay_n)^2 = (x_{n-1} + ay_{n-1})(x_{n+1} + ay_{n+1})$

$\Rightarrow$ $(x_1 + ay_1)^2 = (x_0 + ay_0)(x_2 + ay_2)$ .... (1)

But $x_0 = 1$, $y_0 = 0$ $\Rightarrow$ $x_1 = 2$, $y_1 = 1$ $\Rightarrow$ $x_2 = 7$, $y_2 = 4$ .... (2)

(2) ↓ (1), $(a + 2)^2 = [1 + a(0)]\,[7 + a(4)] = 7 + 4a \quad \Rightarrow \quad a = \pm\sqrt{3}$ .... (3)

(ii) The gradient of the straight line $OP_n = \frac{y_n}{x_n} = \frac{2x_{n-1} + 3y_{n-1}}{x_{n-1} + 2y_{n-1}} = \frac{2 + 3\frac{y_{n-1}}{x_{n-1}}}{1 + 2\frac{y_{n-1}}{x_{n-1}}}$ …. (4)

Putting $m = \lim_{n\to\infty} \frac{y_{n-1}}{x_{n-1}} = \lim_{n\to\infty} \frac{y_n}{x_n}$ and taking $n \to \infty$ in (4), (existence of limit not proved) we have

$m = \frac{2 + 3m}{1 + 2m} \Rightarrow m^2 - m - 1 = 0 \Rightarrow m = \frac{1 + \sqrt{5}}{2}$, m is positive since $x_n, y_n > 0$.

11. $a_{n+1} - \sqrt{5} = \frac{a_n^{\ 2} + 5}{2a_n} - \sqrt{5} = \frac{a_n^{\ 2} - 2\sqrt{5}a_n + 5}{2a_n} = \frac{(a_n - 5)^2}{2a_n} > 0 \Rightarrow a_{n+1} > \sqrt{5} \quad \Rightarrow a_n > \sqrt{5}$

$a_{n+1} - \sqrt{5} = \frac{(a_n - 5)^2}{2a_n} < \frac{(a_n - 5)^2}{2\sqrt{5}}$ …. (1)

Let $P(n)$ be the proposition $a_{n+1} - \sqrt{5} < \frac{\left(3 - \sqrt{5}\right)^{2^n}}{\left(2\sqrt{5}\right)^{2^n - 1}}$.

$P(1)$ is true as $a_1 - \sqrt{5} = 3 - \sqrt{5} < \frac{3 - \sqrt{5}}{2\sqrt{5} - 1}$.

Assume $P(k - 1)$ is true for some $k \in \mathbb{N}$, i.e. $a_k - \sqrt{5} < \frac{\left(3 - \sqrt{5}\right)^{2^{k-1}}}{\left(2\sqrt{5}\right)^{2^{k-1} - 1}}$ …. (2)

For $P(k)$, $a_{k+1} - \sqrt{5} < \frac{(a_k - 5)^2}{2\sqrt{5}} < \left(\frac{\left(3 - \sqrt{5}\right)^{2^{k-1}}}{\left(2\sqrt{5}\right)^{2^{k-1} - 1}}\right)^2 \frac{1}{2\sqrt{5}}$, by (2)

$$= \frac{\left(3 - \sqrt{5}\right)^{2^k}}{\left(2\sqrt{5}\right)^{2^k - 1}} \qquad \therefore \quad P(k + 1) \text{ is true .}$$

By the Principle of Mathematical Induction, $P(n)$ is true, $\forall n \in \mathbb{N}$.

$$0 < a_{n+1} - \sqrt{5} < \frac{\left(3 - \sqrt{5}\right)^{2^n}}{\left(2\sqrt{5}\right)^{2^n - 1}} = \left(2\sqrt{5}\right)\left(\frac{3 - \sqrt{5}}{2\sqrt{5}} \times \frac{3 + \sqrt{5}}{3 + \sqrt{5}}\right)^{2^n} < 6\left(\frac{2}{5 + 3\sqrt{5}}\right) < 6 \times \left(\frac{2}{11}\right)^{2^n}$$

By Sandwich theorem, $\lim_{n\to\infty} 6 \times \left(\frac{2}{11}\right)^{2^n} = 0 \Rightarrow \lim_{n\to\infty}\left(a_{n+1} - \sqrt{5}\right) = 0 \Rightarrow \lim_{n\to\infty} a_{n+1} = \sqrt{5} = \lim_{n\to\infty} a_n$

12. If r_n converges, Let $L = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{n+1}$

Take $n\to\infty$ in $r_{n+1} + \frac{1}{r_n} = 2A \quad \Rightarrow L + \frac{1}{L} = 2A \quad \Rightarrow L^2 - 2AL + 1 = 0$ …. (1)

L has real value if $\Delta \ge 0$ or $A \ge 1 \quad \Rightarrow$ The necessary condition for convergence is $A \ge 1$.

Let α, β be the roots of the equation (1), where $\alpha < \beta$.

$$\alpha + \beta = 2A, \quad \alpha\beta = 1, \quad \alpha = A - \sqrt{A^2 - 1} < A, \qquad \beta = A + \sqrt{A^2 - 1} > A \qquad \ldots. \quad (2)$$

$$r_{n+1} + \frac{1}{r_n} = 2A \Rightarrow r_{n+1} + \frac{\alpha\beta}{r_n} = \alpha + \beta \Rightarrow r_{n+1} - (\alpha + \beta) + \frac{\alpha\beta}{r_n} = 0 \qquad \ldots. \quad (3)$$

(i) If $r_0 = \alpha$, then $r_n = \alpha \quad \forall n \in \mathbb{N}$ and $r_n \to \alpha$.

(ii) If $r_0 \neq \alpha$, then from (3), $r_{n+1} - \alpha = \frac{\beta}{r_n}(r_n - \alpha)$ …. (4) and $r_{n+1} - \beta = \frac{\alpha}{r_n}(r_n - \beta)$ …. (5)

$\frac{(5)}{(4)}$, $\quad \frac{r_{n+1} - \beta}{r_{n+1} - \alpha} = \frac{\alpha}{\beta}\frac{r_n - \beta}{r_n - \alpha} = \left(\frac{\alpha}{\beta}\right)^2 \frac{r_{n-1} - \beta}{r_{n-1} - \alpha} = \ldots = \left(\frac{\alpha}{\beta}\right)^{n+1} \frac{r_0 - \beta}{r_0 - \alpha}$

Solve for r_{n+1}, we have $r_{n+1} = \dfrac{\beta - \alpha\left(\frac{\alpha}{\beta}\right)^{n+1}\frac{r_0 - \beta}{r_0 - \alpha}}{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}\frac{r_0 - \beta}{r_0 - \alpha}}$ …. (6)

By (2), $\frac{\alpha}{\beta} \leq 1$. **(a)** $\frac{\alpha}{\beta} < 1 \Rightarrow \left(\frac{\alpha}{\beta}\right)^{n+1} \to 0$ as $n \to \infty$. From (6), $r_n \to \beta$ for any r_0.

(b) If $\frac{\alpha}{\beta} = 1$, then from (2), $\alpha = \beta = A = 1$, $r_{n+1} + \frac{1}{r_n} = 2$.

Let $P(n)$ be the proposition $r_n = \dfrac{(n+1)r_0 - n}{nr_0 - (n-1)}, \forall n \in \mathbb{N} \cup \{0\}$.

$P(0)$ is obviously true.

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. $r_k = \dfrac{(k+1)r_0 - k}{kr_0 - (k-1)}$ …. (7)

For $P(k+1)$, , $r_{k+1} = 2 - \dfrac{1}{r_k} = 2 - \dfrac{kr_0 - (k-1)}{(k+1)r_0 - k} = \dfrac{(k+2)r_0 - (k+1)}{(k+1)r_0 - k}$, by (7)

$\therefore$ $P(k+1)$ is true . By the Principle of Mathematical Induction, $P(n)$ is true , $\forall n \in \mathbb{N}$.

Now, $r_n = \dfrac{(n+1)r_0 - n}{nr_0 - (n-1)} = \dfrac{n(r_0 - 1) + r_0}{n(r_0 - 1) + 1}$

(i) If $r_0 \neq 1$, $r_n = \dfrac{(r_0 - 1) + (r_0/n)}{(r_0 - 1) + (1/n)} \to \dfrac{r_0 - 1}{r_0 - 1} = 1$. **(ii)** If $r_0 = 1$, $r_n = r_0 \to 1$.

$\therefore$ The sufficient condition for convergence is $A \geq 1$.

13. Let $u_n = \sin\frac{n\pi}{5}$. Consider two subsequences,

$$u_{10k} = \sin\frac{10k\pi}{5} = \sin 2k\pi = 0, \qquad u_{10k+1} = \sin\frac{(10k+1)\pi}{5} = \sin\left(2k\pi + \frac{\pi}{5}\right) = \sin\frac{\pi}{5}$$

$\therefore \quad u_{10k} \to 0, \qquad u_{10k+1} \to \sin\frac{\pi}{5} \neq 0$

u_n is divergent because there exists two subsequences having different limits.

14. (Using Cauchy convergent criterion)

$$u_n = 1 + \frac{1}{3} + \ldots + \frac{1}{2n-1},$$

$$\left|u_{n+p} - u_n\right| = \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{2(n+p)-1} > \frac{1}{2(n+p)-1} + \ldots + \frac{1}{2(n+p)-1} = \frac{p}{2(n+p)-1}$$

Take $p = n$, $\left|u_{2n} - u_n\right| > \frac{n}{4n-1} > \frac{n}{4n} = \frac{1}{4}$

Take $\varepsilon = \frac{1}{4}$, there does not exist an n such that $\left|u_{2n} - u_n\right| < \varepsilon$.

(Not so formal) $S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \ldots . = \left(1 + \frac{1}{3}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{9} + \frac{1}{11}\right) + \ldots . > 1 + \frac{1}{3} + \frac{1}{5} + \ldots = S$

Contradiction.

15. For the first part, please see Limit of sequences Set 1 No. 7, 8.

Given that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$. Take $a_n = \left(1 + \frac{1}{n}\right)^n$, then

$$\left(a_1 a_2 \ldots a_n\right)^{1/n} = \left[2^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = \left[\frac{1}{2}\frac{1}{3}\ldots\frac{1}{n}(n+1)^n\right]^{1/n} = \frac{n+1}{[n!]^{1/n}}$$

$$\therefore \quad \lim_{n\to\infty}\frac{n+1}{[n!]^{1/n}} = e \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}(n!)^{1/n} = \lim_{n\to\infty}\frac{n+1}{n} \Big/ \lim_{n\to\infty}\frac{n+1}{[n!]^{1/n}} = \frac{1}{e}$$

Take $a_n = n$, $a_{n+1} = n + 1$, then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{n} = 1$, $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty} n^{1/n} = 1$.

16. Same as Limit of Sequences Set 1 No. 24.

17. The first part is same as Limit of Sequences Set 1 , No. 35, by taking $A = \sqrt{a}$.

The case that $x_0 > \sqrt{a} > 0$ to prove that $\sqrt{a} < x_n < x_{n-1} < \ldots < x_0$ is also given in that question.

If $x_0 = \sqrt{a}$, then $\sqrt{a} = x_n = x_{n-1} = \ldots = x_0$.

x_n does not converge if $a < 0$, since if it converges, then let $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \Rightarrow L = \frac{1}{2}\left(L + \frac{a}{L}\right) \Rightarrow L^2 = a$$ which has no real solution for $a < 0$.

18. $$x_n - x_{n-1} = \frac{x_{n-1} + x_{n-2}}{2} - x_{n-1} = -\frac{1}{2}\left(x_{n-1} - x_{n-2}\right) = \left(-\frac{1}{2}\right)^2\left(x_{n-2} - x_{n-3}\right) = \ldots = \left(-\frac{1}{2}\right)^{n-1}\left(x_1 - x_0\right)$$

$$\sum_{r=1}^{n}\left(x_r - x_{r-1}\right) = \left(x_1 - x_0\right)\sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1} \Rightarrow x_n - x_0 = \left(x_1 - x_0\right)\frac{1 - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}\left(x_1 - x_0\right)\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

$$\therefore \quad x_n = \frac{2}{3}\left(x_1 - x_0\right)\left[1 - \left(-\frac{1}{2}\right)^n\right] + x_0 \Rightarrow \qquad x_n \to \frac{2}{3}\left(x_1 - x_0\right) + x_0 = \frac{1}{3}\left(x_0 + 2x_1\right)$$, as $n \to \infty$.

19. $\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}\Rightarrow\lim_{n\to\infty}\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^n}=\lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}=\lim_{n\to\infty}\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)}\frac{\sin x}{x}=\frac{\sin x}{x}$

Take $\lambda=\mu\cos x$. We can use Mathematical Induction to prove that

P(n) : $x_n=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{n-1}}\right]\cos\frac{x}{2^{n-1}}$, $y_n=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{n-1}}\right]$, $n\geq 2$.

For P(2), $x_2=\frac{x_1+y_1}{2}=\frac{1}{2}[\lambda+\mu]=\frac{1}{2}[\mu\cos x+\mu]=\frac{\mu}{2}[\cos x+1]=\mu\cos^2\frac{x}{2}$

$y_2=\sqrt{x_2y_1}=\sqrt{\mu\cos^2\frac{x}{2}\mu}=\cos\frac{x}{2}$ $\therefore$ P(2) is true.

Assume P(k) is true for some $k\in\mathbb{N}$, $k\geq 2$, i.e.

$x_k=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\cos\frac{x}{2^{k-1}}$, $y_k=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]$ …. (1)

For P(k + 1), $x_{k+1}=\frac{x_k+y_k}{2}=\frac{1}{2}\left\{\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\cos\frac{x}{2^{k-1}}+\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\right\}$

$=\frac{1}{2}\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\left\{\cos\frac{x}{2^{k-1}}+1\right\}=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\cos^2\frac{x}{2^k}$

$=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^k}\right]\cos\frac{x}{2^k}$

$y_{k+1}=\sqrt{x_{k+1}y_k}=\sqrt{\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^k}\right]\cos\frac{x}{2^k}\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]}$

$=\mu\left[\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{k-1}}\right]\cos\frac{x}{2^k}$ $\therefore$ P(k + 1) is true .

By the Principle of Mathematical Induction, P(n) is true , $\forall n\in\mathbb{N}\setminus\{1\}$.

Both x_n and y_n tend to $\mu\frac{\sin x}{x}=\frac{\sqrt{\mu^2-\lambda^2}}{\cos^{-1}\left(\frac{\lambda}{\mu}\right)}$ since $\lambda=\mu\cos x$.

20. Let P(n) be the proposition $\left(\sqrt{3}+1\right)^n=a_n\sqrt{3}+b_n$, $\forall n\in\mathbb{N}\cup\{0\}$.

P(1) is true since $\left(\sqrt{3}+1\right)^1=a_1\sqrt{3}+b_1$, $a_1=b_1=1$

Assume P(k) is true for some $k\in\mathbb{N}$, i.e. $\left(\sqrt{3}+1\right)^k=a_k\sqrt{3}+b_k$ …. (1)

For P(k+1), , $\left(\sqrt{3}+1\right)^{k+1}=\left(\sqrt{3}+1\right)^k\left(\sqrt{3}+1\right)=\left(a_k\sqrt{3}+b_k\right)\left(\sqrt{3}+1\right)$, by (1)

$=\left(a_k+b_k\right)\sqrt{3}+\left(3a_k+b_k\right)=\left(a_{k+1}\sqrt{3}+b_{k+1}\right)$

$\therefore$ P(k + 1) is true . By the Principle of Mathematical Induction, P(n) is true , $\forall n\in\mathbb{N}$.

(a) $a_{n+2}\sqrt{3}+b_{n+2}=\left(\sqrt{3}+1\right)^{n+2}=\left(\sqrt{3}+1\right)^{n+1}\left(\sqrt{3}+1\right)=\left(a_{n+1}\sqrt{3}+b_{n+1}\right)\left(\sqrt{3}+1\right)$

$=\left(a_{n+1}+b_{n+1}\right)\sqrt{3}+\left(3a_{n+1}+b_{n+1}\right)$

$\therefore\begin{cases}a_{n+2}=a_{n+1}+b_{n+1}\\b_{n+2}=3a_{n+1}+b_{n+1}\end{cases}\Rightarrow\begin{cases}a_{n+2}=2a_{n+1}+b_{n+1}-a_{n+1}=2a_{n+1}+\left(3a_n+b_n\right)-\left(a_n+b_n\right)=2\left(a_{n+1}+a_n\right)\\b_{n+2}=2b_{n+1}+3a_{n+1}-b_{n+1}=2b_{n+1}+3\left(a_n+b_n\right)-\left(3a_n+b_n\right)=2\left(b_{n+1}+b_n\right)\end{cases}$

(b) Use Mathematical Induction , proof not given.

(c) $\left(\sqrt{3}+1\right)^n\left(\sqrt{3}-1\right)^n=(3-1)^n \Rightarrow \left(a_n\sqrt{3}+b_n\right)(-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)=2^n \Rightarrow 3a_n{}^2-b_n{}^2=(-1)^{n-1}2^n$

(d) From $\left(\sqrt{3}+1\right)^n=a_n\sqrt{3}+b_n$ and $\left(\sqrt{3}-1\right)^n=(-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)$, solving, we have

$$a_n=\frac{1}{2\sqrt{3}}\left[\left(\sqrt{3}+1\right)^n-\left(\sqrt{3}+1\right)^n\right], \qquad b_n=\frac{1}{2}\left[\left(\sqrt{3}+1\right)^n+\left(\sqrt{3}+1\right)^n\right]$$

$$\frac{b_n}{a_n}=\frac{\frac{1}{2}\left[\left(\sqrt{3}+1\right)^n+\left(\sqrt{3}+1\right)^n\right]}{\frac{1}{2\sqrt{3}}\left[\left(\sqrt{3}+1\right)^n-\left(\sqrt{3}+1\right)^n\right]}=\sqrt{3}\,\frac{1+\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^n}{1-\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^n}\Rightarrow \lim_{n\to\infty}\frac{b_n}{a_n}=\sqrt{3}$$

21. **(a)** From No. 19, $P_n=\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\frac{x}{2^n}}$. $\lim_{x\to\pi/2}\left(\lim_{n\to\infty}P_n\right)=\lim_{x\to\pi/2}\frac{\sin x}{x}=\frac{\sin(\pi/2)}{(\pi/2)}=\frac{2}{\pi}$

(b) $\lim_{n\to\infty}\cos nx=0\Rightarrow\lim_{n\to\infty}\cos 2nx=0\Rightarrow\lim_{n\to\infty}\left(2\cos^2 nx-1\right)=0\Rightarrow\lim_{n\to\infty}\cos nx=\pm\sqrt{\frac{1}{2}}\Rightarrow$ contradiction

(c) sin (n+1) x – sin (n-1) x = 2 cos nx sin x

$\lim_{n\to\infty}\sin nx=0\Leftrightarrow\lim_{n\to\infty}\sin(n+1)x=\lim_{n\to\infty}\sin(n-1)x=0\Leftrightarrow\lim_{n\to\infty}\cos nx\sin x=0\Leftrightarrow\sin x=0$, by **(b) .**

$\Leftrightarrow$ $x = m\pi$, where $m\in\mathbb{Z}$.

22. Let P(n) be the proposition: $(1 + x)^n \geq 1 + nx$ $\forall n\in\mathbf{N}$ and $\forall x\in\mathbf{R}$, $x \geq -1$.

For P(1), $(1 + x)^1 = 1 \geq 1 + x$ $\therefore$ P(1) is true.

Assume P(k) is true for some $k\in\mathbf{Z}$, $k \geq 0$,

that is, $(1 + x)^k \geq 1 + kx$ $\forall\, k\in\mathbf{N}\cup\{0\}$ and $\forall x\in\mathbf{R}$, $x \geq -1$. (1)

For P(k+1), $(1 + x)^{k+1} = (1 + x)^k(1 + x)$

$\geq (1 + kx)(1 + x)$, by (1) and also note that since $x \geq -1$, the factor $(x + 1) > 0$.

$= 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$

$\therefore$ P(k+1) is also true. By the Principle of Mathematical Induction, P(n) is true $\forall n\in\mathbb{N}$.

(a) $t > 1 \Leftrightarrow \ln t > 0 \Leftrightarrow (1/n)\ln t > 0 \quad \forall n > 0 \Leftrightarrow \sqrt[n]{t} > 1$.

(b) Putting $\sqrt[n]{t} = 1 + x_n$. Since $\sqrt[n]{t} > 1$. $\therefore$ $x_n > 0$ $\forall n > 0$.

By **(a)**, $(1 + x_n)^n > 1 + nx_n$.

$\therefore$ $t > 1 + nx_n \Rightarrow \frac{t-1}{n} > x_n$

$1 < \sqrt[n]{t} = 1 + nx_n < 1+\frac{t-1}{n}$ …. (2)

For $t > 1$, taking $n\to\infty$ in (2) , $1\leq\lim_{n\to\infty}\left(1+x_n\right)\leq\lim_{n\to\infty}\left(1+\frac{t-1}{n}\right)=1$

By Sandwich Theorem, $\lim_{n\to\infty}\left(1+x_n\right)=1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{t}=1$.